1 The heat equation.

We solve the boundary eigenvalue problem[1]:

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - k\,\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad 0 < x < l, \quad t > 0,\\
&u(0,t) = u(l,t) = 0, \quad t > 0,\\
&u(x,0) = x,
\end{aligned}$$

where k and l are physical constants. By the separation of variables, we assume a solution of the form $u(x,t) = X(x)T(t)$, where $X(x)$ represents the spatial and $T(t)$ the temporal component, respectively. Upon substitution in the equation we have

$$\frac{\dot{T}(t)}{T(t)} = k\,\frac{X''(x)}{X(x)} = -\lambda,$$

where the functions are equal only if both are equal to some constant denoted by $-\lambda$. The minus sign is for convenience as we seek an imaginary solution that is convertible to a series of sines and cosines. Generally one chooses the spatial ordinary differential equation first because its eigenfunctions will be more fundamental in the resulting separation of variables solution.

Rearranging in a more familiar form, the ODE is

$$X''(x) = \frac{-\lambda}{k}\,X(x).$$

Letting $D = \frac{d}{dx}$ and substituting back into the equation, we have

$$D^2 = \frac{-\lambda}{k},$$

or

$$D = \pm\sqrt{\frac{-\lambda}{k}} = \pm\, i\,\sqrt{\frac{\lambda}{k}}.$$

Then

$$\begin{aligned}
X(x) &= c_1' e^{i\sqrt{\frac{\lambda}{k}}\,x} + c_2' e^{-i\sqrt{\frac{\lambda}{k}}\,x}\\
&= c_1'\left(\cos\sqrt{\frac{\lambda}{k}}\,x + i\,\sin\sqrt{\frac{\lambda}{k}}\,x\right) + c_2'\left(\cos\sqrt{\frac{\lambda}{k}}\,x - i\,\sin\sqrt{\frac{\lambda}{k}}\,x,\right)
\end{aligned}$$

where c_1' and c_2' denote arbitrary imaginary constants. Multiplying through and combining, the solution is

$$X(x) = c_1 \cos\sqrt{\frac{\lambda}{k}}\,x + c_2 \sin\sqrt{\frac{\lambda}{k}}\,x,$$

where $c_1 = c_1' + c_2'$ and $c_2 = i(c_1' - c_2')$.

[1] See Gustafson pp. 38-9.

Applying the boundary conditions at $u(0)$ and $u(l)$ we have

$$0 = c_1$$

and

$$0 = c_2 \sin \sqrt{\frac{\lambda}{k}}\, l$$

or

$$\lambda_n = \frac{n^2\pi^2}{l^2}\, k,$$

where the eigenvalues correspond to integers $n = 1, 2, 3, ...$, and where the arcsine of zero is equal to $\pm n\pi$. These acquired eigenvalues λ_n determine the temporal factors which we immediately integrate to get:

$$\frac{dT(t)}{T(t)} = -\lambda T(t),$$

hence

$$T_n(t) = d_n e^{-\lambda_n t}.$$

Substituting the results:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2 k}{l^2}\, t}, \tag{1}$$

where the constant d_n was absorbed into c_n. Forming the linear superposition provides our formal separation of variables solution. Applying the third boundary condition ($u(x, 0)$), we have

$$x = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2 k}{l^2}\, t}.$$

Solving for the Fourier coefficients we multiply both sides by $\sin \frac{m\pi x}{l}$ and integrate (because if the functions are equal, the integral of the functions are equal too) to get

$$\int_0^l x\, \sin \frac{m\pi x}{l}\, dx = \int_0^l c_n\, sin\frac{n\pi x}{l}\, sin\frac{m\pi x}{l}\, dx.$$

Recall the right-hand side (RHS) integral is zero if $m \neq n$ and not zero if $m = n$. Letting $m = n$ we have

$$\int_0^l x\, \sin \frac{n\pi x}{l}\, dx = \int_0^l c_n\, sin^2\frac{n\pi x}{l} dx.$$

A simple way of evaluating the RHS is to observe that $\frac{\sin^2\theta + \cos^2\theta}{2} = 1$. Adding $\cos^2 \frac{n\pi x}{l}$ and dividing by 2, the RHS integral reduces to

$$c_n \int_0^l \frac{1}{2}\, dx.$$

Integrating the RHS and solving for the Fourier coefficient

$$c_n = \frac{2}{l} \int_0^l x\, \sin \frac{n\pi x}{l}\, dx.$$

Integrating by parts we let $u = x$, $du = dx$, $dv = \sin \frac{n\pi x}{l}$, and $v = -\frac{l}{n\pi} \cos \frac{n\pi x}{l}$. Then

$$c_n = \frac{2}{l}\left[-\frac{xl}{n\pi} \cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2} \sin\frac{n\pi x}{l}\right]\Bigg|_0^l$$

$$= \frac{-2l}{n\pi}\cos n\pi = \frac{2l}{n\pi}(-1)^{n+1}.$$

The final solution is then

$$u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} \, e^{-\frac{n^2\pi^2 k}{l^2} t}. \tag{2}$$

Note that the physical scales are reflected in the eigenfunctions, decay rates, and Fourier coefficients of the solution.